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LETTER TO THE EDITOR

Method of a two-dimensional equation for the three-dimensional problem of electromagnetic wave propagation in a thin inhomogeneous waveguide

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Abstract. A method for analysis of a three-dimensional radiowave propagation problem in a thin spherical inhomogeneous waveguide is presented. This method has a wider range of applications than the technique of telegraphist's equations. The solution of the three-dimensional boundary value problem is approximated by means of the solution of a two-dimensional Schrödinger equation with a periodic potential.

The solution of the three-dimensional problem of radiowave propagation in an inhomogeneous medium involves certain analytical and numerical difficulties. An idea to decrease the dimension of the problem can appreciably help to simplify the determination of the solution. The technique of telegraphist's equations [1, 2] can be considered as an example of the implementation of this idea.

The reduction of the problem to fewer dimensions, however, may lead to a loss of some essential three-dimensional properties of the solution [3]. For instance, the solution of an inhomogeneous problem can be represented by expansion in terms of eigenfunctions which are solutions of the corresponding homogeneous one [4]. This representation is natural under validity conditions of telegraphist's equations [5]. The inhomogeneity of a medium may result in the appearance of new eigenvalues in the problem, and, consequently, in an increase of dimensionality of a subspace of eigenfunctions. In this case the above technique does not function (see in detail [5] section 35).

The goal of the present paper is, proceeding from a complete set of three-dimensional Maxwell equations, to derive two-dimensional differential equations that could embrace the largest possible circle of physical effects resulting from the inhomogeneity of the medium. The characteristic dimension of a waveguide along one of the coordinates is assumed to be less than a wavelength (a thin waveguide).

Let us consider a concrete physical system. Let a domain represent a spherical layer (Earth–ionosphere cavity) with a well-conducting Earth and an inhomogeneous ionosphere.

The vector wave equations for the electric field $\boldsymbol{E}$ derived from Maxwell's equations in a spherical coordinate frame (r, θ, φ) can be written as

$$\frac{\partial^2 E_r}{\partial r^2}+\frac{4}{r}\frac{\partial E_r}{\partial r}+\frac{1}{r^2}\left(\cot\theta\frac{\partial E_r}{\partial\theta}+\frac{\partial^2 E_r}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2 E_r}{\partial\varphi^2}\right)+\left(\frac{\omega^2}{c^2}+\frac{2}{r^2}\right)E_r=0 \qquad (1a)$$

$$\frac{\partial^2 E_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial E_\theta}{\partial r}+\frac{1}{r^2}\left(3\cot\theta\frac{\partial E_\theta}{\partial\theta}+\frac{\partial^2 E_\theta}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2 E_\theta}{\partial\varphi^2}\right)+\left(\frac{\omega^2}{c^2}-\frac{2}{r^2}+\frac{1}{r^2\sin^2\theta}\right)E_\theta$$
$$=-\frac{2}{r}\left(\cot\theta\frac{\partial E_r}{\partial r}+\frac{2}{r}\cot\theta E_r+\frac{1}{r}\frac{\partial E_r}{\partial\theta}\right) \tag{1b}$$

$$\frac{\partial^2 E_\varphi}{\partial r^2}+\frac{2}{r}\frac{\partial E_\varphi}{\partial r}+\frac{1}{r^2}\left(\cot\theta\frac{\partial E_\varphi}{\partial\theta}+\partial^2 E_\varphi+\frac{1}{\sin^2\theta}\frac{\partial^2 E_\varphi}{\partial\varphi^2}\right)+\left(\frac{\omega^2}{c^2}-\frac{1}{r^2\sin^2\theta}\right)E_\varphi$$
$$=-\frac{2}{r^2\sin\theta}\left(\cot\theta\frac{\partial E_\theta}{\partial\varphi}+\frac{\partial E_r}{\partial\varphi}\right). \tag{1c}$$

For simplicity, the present letter makes use of the following boundary conditions:

$$E_\theta|_0=E_\varphi|_0=0 \tag{2a}$$

$$E_\theta|_1=z(\theta,\varphi)H_\varphi|_1 \qquad E_\varphi|_1=-z(\theta,\varphi)H_\theta|_1. \tag{2b}$$

Subscripts 0 and 1 indicate the magnitude of functions E_θ, E_φ, H_θ and H_φ at the surface of the Earth and ionosphere, respectively. Distance h between the Earth and ionosphere is set constant. The ionosphere is characterized by surface impedance $z(\theta,\varphi)$.

The set of equations (1) is given in a somewhat unusual, but convenient form for our goals: equation $(1a)$ is not bound to the other equations of the set. The right-hand part of equation $(1b)$ is determined by the solution of $(1a)$. The right-hand part of the equation $(1c)$ is determined by the solutions of equations $(1a)$ and $(1b)$.

Owing to good conductivity of the Earth and ionosphere the components E_θ and E_φ are small as compared with E_r. Therefore, first of all, we shall focus our attention on equation $(1a)$ which we shall write as follows:

$$\begin{gathered}\frac{\partial^2 \tilde{E}_r}{\partial r^2}+\frac{2}{r}\frac{\partial \tilde{E}_r}{\partial r}+\frac{1}{r^2}\left(\cot\theta\frac{\partial^2 \tilde{E}_r}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2 \tilde{E}_r}{\partial\varphi^2}\right)+\frac{\omega^2}{c^2}\tilde{E}_r=0\\ \tilde{E}_r=rE_r.\end{gathered} \tag{3}$$

Together with variables r, θ, φ it will be convenient to use variables x, θ, φ, where $x=(r-r_0)/h$, r_0 is the radius of the Earth, $h=r_1-r_0$, r_1 being the radius of the ionosphere. We shall put magnitude $z=a\delta(\theta,\varphi)$ small: $a\leqslant h/r_0\ll 1$, (a being a small constant value); however, $\partial\delta/\partial\theta\leqslant 1$ and $\partial\delta/\partial\varphi\leqslant 1$, i.e. dependence z on θ and φ can be fast. The desired fields, though, are assumed to be fairly smooth.

Let us divide a fast and slow dependence over variable r, writing $\tilde{E}_r$ in the form of

$$\tilde{E}_r=R(r;\theta,\varphi)U(\theta,\varphi;x) \tag{4}$$

where R is solution of the following problem:

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\kappa^2-\frac{\sigma}{r^2}\right)R=0 \tag{5a}$$

$$\left.\frac{dR}{dr}\right|_1=-\mathrm{i}\kappa z(\theta,\varphi)R\Big|_1 \qquad \left.\frac{dR}{dr}\right|_0=0. \tag{5b}$$

σ is a number (a complex one in the general case). The solution of the problem (5) has been well studied (it is a linear combination of Bessel functions, for example), therefore we shall consider function $R(r;\theta,\varphi)$ as a well known one.

Thus, the problem (5) gives us a set of basis functions which parametrically depend on θ, φ ($\kappa=\kappa(\theta,\varphi)$ if z depends on θ, φ).

Function $U(\theta,\varphi;x)$ will be presented as

$$U=\sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{\partial^n U}{\partial x^n}\bigg|_0. \tag{6}$$

Making use of the equation div $\boldsymbol{E}=0$ and of boundary conditions of $(2a)$, we obtain

$$\frac{\partial U}{\partial x}\bigg|_0=-\frac{h}{r_0}U_0 \qquad U_0\equiv U(\theta,\varphi;0). \tag{7}$$

Employing equation (3) and the boundary conditions, we have:

$$\begin{gathered}\frac{\partial^2 U_0}{\partial x^2}\bigg|_0=-\frac{h^2}{r_0^2}\left(\frac{\partial^2 U_0}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2 U_0}{\partial\varphi^2}+\cot\theta\frac{\partial U_0}{\partial\theta}+\tilde{\sigma}U_0-2U_0\right)\\ \tilde{\sigma}=r_0^2(\omega^2/c^2-\kappa^2)+\sigma.\end{gathered} \tag{8}$$

One has to take into account the terms $\partial R/\partial\varphi$, $\partial R/\partial\theta$, $\partial^2 R/\partial\varphi^2,\ldots$ in equation (8) and below when the amplitude a is not small.

In such a way we can calculate a derivative of any order of magnitude, entering in relation (6). All of them are expressed through function U_0 and its derivatives over θ and φ. Here we shall confine ourselves to terms of the order of h^2/r_0^2.

Our task now is to find an equation for U_0. Let us use for this goal the boundary conditions on the ionosphere $(2b)$ and div $\boldsymbol{E}=0$. We have

$$\frac{\partial U}{\partial x}\bigg|_1+\left[\frac{h}{r_1}+\mathrm{i}h\left(\frac{\omega}{c}-\kappa\right)z\right]U_1-\frac{\mathrm{i}c}{\omega r_0}\frac{h}{r_0}\left(\frac{\partial z}{\partial\theta}\frac{\partial U}{\partial\theta}\bigg|_1+\frac{1}{\sin^2\theta}\frac{\partial z}{\partial\varphi}\frac{\partial U}{\partial\varphi}\bigg|_1\right)=0. \tag{9}$$

Let us make an expansion in (9) over powers of h/r_0, using (7), (8) and equalling the terms with the same powers of h/r_0. Linear terms over h/r_0 in (9) are reduced. Quadratic terms over h/r_0 yield the equation

$$\begin{aligned}\frac{\partial^2 U_0}{\partial\theta^2}+\cot\theta\frac{\partial U_0}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 U_0}{\partial\varphi^2}+\frac{\mathrm{i}c}{\omega}\frac{a}{h}\left(\frac{\partial\delta}{\partial\theta}\frac{\partial U_0}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial\delta}{\partial\varphi}\frac{\partial U_0}{\partial\varphi}\right)&\\ +[r_0^2(\omega^2/c^2-\kappa^2)-\mathrm{i}r_0^2(\omega/e-\kappa)z/h+\sigma]U_0=0.&\end{aligned} \tag{10}$$

We note that the factor for the terms resulting from inhomogeneity of the medium, in the framework of the assumptions made, is $ca/(\omega h)\leqslant 1$. If $\delta=$ constant, from (10) there follow well known solutions for a homogeneous cavity ($\kappa=\omega/c$, $\sigma=n(n+1)$).

It is natural that the equations obtained at different powers of h/r_0 do not contradict one another. This is clear from the scheme of obtaining these equations. A rigorous proof of this fact is omitted here.

The equation (10) can be reduced to the form of the Schrödinger equation with a periodic potential. For this we shall make the replacement

$$U_0=u\exp\Phi \qquad \Phi=-\frac{\mathrm{i}}{2}\frac{ca}{\omega h}\delta$$

then for u we shall have the equation:

$$-\Delta u+W(\theta,\varphi)u=\sigma u. \tag{11}$$

Here

$$\Delta = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$(\nabla\cdot)^2 = \left(\frac{\partial}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\varphi}\right)^2$$

$$W = \frac{\mathrm{i}}{2}\frac{ca}{\omega h}\Delta\delta - \frac{1}{4}\left(\frac{ca}{\omega h}\right)^2(\nabla\delta)^2 - [r_0^2(\omega^2/c^2-\kappa^2) - \mathrm{i}r_0^2(\omega/c-\kappa)z/h].$$

The potential $W(\theta, \varphi)$ is a periodic function of φ. At the 'boundary' $\theta = 0$, π, u is constant, that is determined by a source in the general case. Here, for simplicity, the sources were not taken into account, and equation (11) (or (10)) can be used for studying the spectrum of eigenoscillations of an inhomogeneous cavity.

If the solution of the problem (11) is determined, for example, from numerical experiment, we can, using the formulae (6), (7) and (8), calculate the $U(\theta, \varphi; x)$ function, and then the $E_r(r, \theta, \varphi)$ field as well.

In this letter the main idea of the method has been considered. A scheme of choosing two-dimensional equations, under more realistic boundary conditions on the Earth and ionosphere, or owing for the interaction of guided modes, is not changed in essence and will be published separately.

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